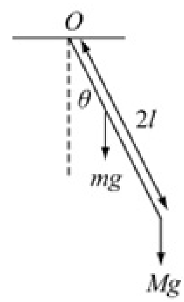
Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 1

Question:

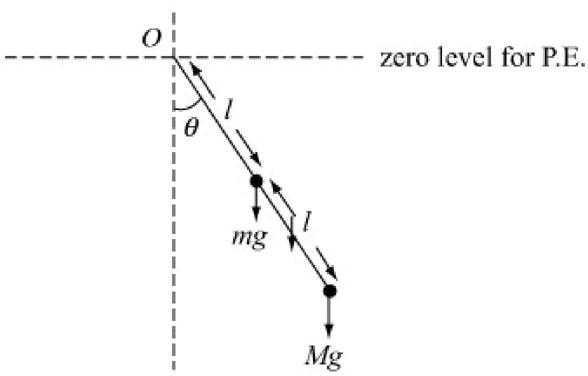
A pendulum is modelled as a uniform rod of mass m and length $2l$ attached to a particle of mass M. The pendulum is smoothly hinged at one end to a fixed point O, as shown in the figure.



a Express the potential energy of the system in terms of θ, the angle which the pendulum makes with the vertical through O.

b Show that there are two positions of equilibrium and determine whether they are stable or unstable.

Solution:



a Take the horizontal level through O as the zero level for potential energy − as O is fixed.

P.E. for rod $= -mgl\cos\theta$

P.E. for particle $= -Mg2l\cos\theta$

$\therefore V = -mgl\cos\theta - 2Mgl\cos\theta$

b $\dfrac{dV}{d\theta} = mgl\sin\theta + 2Mgl\sin\theta$

Put $\dfrac{dV}{d\theta} = 0$. Then $\sin\theta = 0 \Rightarrow \theta = 0$ or π

$\dfrac{d^2V}{d\theta^2} = mgl\cos\theta + 2Mgl\cos\theta$

when $\theta = 0, \dfrac{d^2V}{d\theta^2} = mgl + 2Mgl > 0$ $\therefore$ Equilibrium is stable at the point of minimum potential energy, when $\theta = 0$.

When $\theta = \pi, \dfrac{d^2V}{d\theta^2} = -mgl - 2Mgl < 0$ $\therefore$ Equilibrium is unstable when $\theta = \pi$.

(This is a point of maximum potential energy.)

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

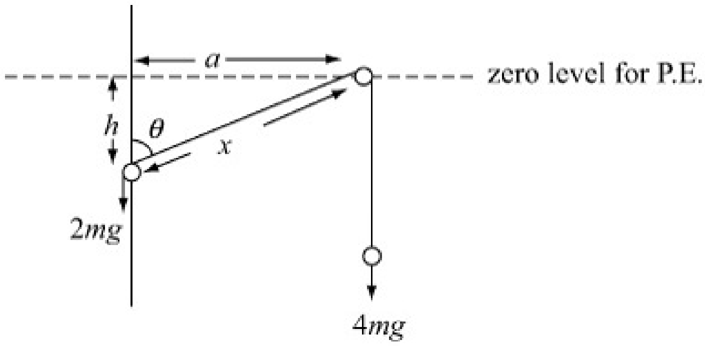
Stability
Exercise A, Question 2

Question:

A small smooth pulley is fixed at a distance a from a fixed smooth vertical wire. A ring of mass $2m$ is free to slide on the wire. It is attached to one end of a string which passes over the pulley and carries a load of mass $4m$ hanging from the other end. The angle between the sloping part of the string and the vertical is θ.

By expressing the potential energy in terms of θ find how far the ring is below the pulley in the equilibrium position and determine whether the equilibrium is stable or unstable.

Solution:



Take the horizontal level through the pulley as the zero level for potential energy as the pulley is fixed.

P.E. for ring $= -2mgh$

But $\tan\theta = \dfrac{a}{h}$, so $h = \dfrac{a}{\tan\theta}$ or $a\cot\theta$ ①

So P.E. for ring $= -2mga\cot\theta$

P.E. for load $= -4mg(l - x)$, where l is the length of the string.

But $\sin\theta = \dfrac{a}{x}$, so $x = \dfrac{a}{\sin\theta}$ or $a\operatorname{cosec}\theta$

$\therefore$P.E. for load $= -4mg(l - a\operatorname{cosec}\theta)$

$\therefore$Total P.E. for system $V = -2mga\cot\theta + 4mga\operatorname{cosec}\theta + k$ where k is constant.

For equilibrium $\dfrac{dV}{d\theta} = 0$

But $\dfrac{dV}{d\theta} = 2mga\operatorname{cosec}^2\theta - 4mga\operatorname{cosec}\theta\cot\theta$ ②

when $\dfrac{dV}{d\theta} = 0$, $\operatorname{cosec}\theta = 0$ or $\cot\theta = \dfrac{1}{2}\operatorname{cosec}\theta$

But $\operatorname{cosec}\theta \neq 0$, for any value of θ

So $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{2\sin\theta}$

$\therefore \cos\theta = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$.

But $h = a\cot\theta = \dfrac{a}{\sqrt{3}}$ (from ①)

i.e. the ring is a distance $\dfrac{a}{\sqrt{3}}$ below the pulley in the equilibrium position.

Differentiate equation ②

$$\frac{d^2V}{d\theta^2} = -4mga\operatorname{cosec}^2\theta\cot\theta + 4mga\operatorname{cosec}^3\theta + 4mga\operatorname{cosec}\theta\cot^2\theta$$

Substitute $\theta = \dfrac{\pi}{3}$, then as $\cot\dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}$ and $\operatorname{cosec}\dfrac{\pi}{3} = \dfrac{2}{\sqrt{3}}$

Then $\dfrac{d^2V}{d\theta^2} = \dfrac{-16mga}{3\sqrt{3}} + \dfrac{32mga}{3\sqrt{3}} + \dfrac{8mga}{3\sqrt{3}} > 0$

$\therefore$ There is a position of stable equilibrium when $h = \dfrac{a}{\sqrt{3}}$.

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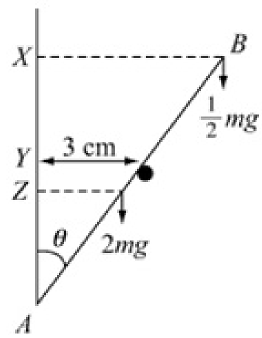
Stability
Exercise A, Question 3

Question:

The diagram shows a uniform rod AB of length 40 cm and mass $2m$ resting with its end A in contact with a smooth vertical wall. The rod is supported by a smooth horizontal rod which is fixed parallel to the wall and a distance 3 cm from the wall as shown in the figure. A particle of mass $\frac{1}{2}m$ is attached to the rod at B.

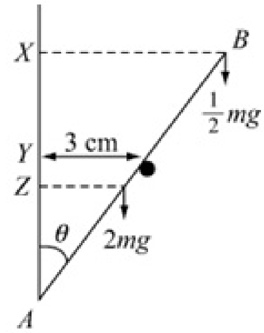


a Show that when AB makes an angle θ with the vertical the potential energy is given by $V = 0.6mg\cos\theta - 0.075mg\cot\theta + \text{constant}$.

b Find any positions of equilibrium and establish whether they are stable or unstable.

Solution:

a Take the horizontal level through the support rod as the zero level for potential energy – as the support rod is fixed



The P.E. for particle $= \frac{1}{2}mg \times XY$

But $XY = AX - AY$

$$= 0.4\cos\theta - \frac{0.03}{\tan\theta}$$

$\therefore$P.E. for particle $= 0.2mg\cos\theta - 0.015mg\cot\theta$

The P.E. for the rod $= -2mg \times YZ$

But $YZ = AY - AZ$

$$= 0.03\cot\theta - 0.2\cos\theta$$

$\therefore$P.E. for rod $= -0.06mg\cot\theta + 0.4mg\cos\theta$

$\therefore$Total P.E. $= V = 0.2mg\cos\theta - 0.015mg\cot\theta - 0.06mg\cot\theta + 0.4mg\cos\theta$

$$= 0.6mg\cos\theta - 0.075mg\cot\theta$$

b $\frac{dV}{d\theta} = -0.6mg\sin\theta + 0.075mg\text{cosec}^2\theta$

Put $\frac{dV}{d\theta} = 0$. Then

$$0.6\sin\theta = \frac{0.075}{\sin^2\theta}$$

$$\therefore \sin^3\theta = \frac{0.075}{0.6} = \frac{1}{8}$$

$$\therefore \sin\theta = \frac{1}{2}$$

So $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ correspond to positions of equilibrium.

$$\frac{d^2V}{d\theta^2} = -0.6mg\cos\theta - 0.15mg\text{cosec}^2\theta\cot\theta$$

when $\theta = \frac{\pi}{6}, \frac{d^2V}{d\theta^2} = -\frac{9\sqrt{3}}{10}mg < 0$ so unstable

when $\theta = \frac{5\pi}{6}, \frac{d^2V}{d\theta^2} = \frac{9\sqrt{3}}{10}mg > 0$ so stable

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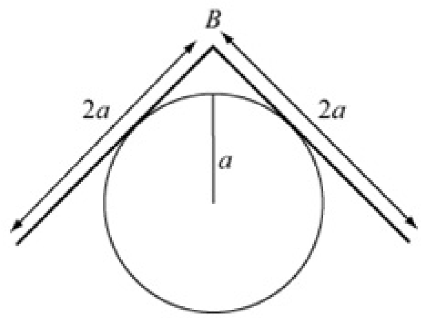
Stability
Exercise A, Question 4

Question:

Two uniform smooth heavy rods, each of mass M and length $2a$, are smoothly jointed together at B. They are placed symmetrically in a vertical plane, over a fixed sphere of radius a as shown.

a Show that when the rods make an angle θ with the horizontal the potential energy V is given by $V = 2Mga(\sec\theta - \sin\theta) + \text{constant}$.

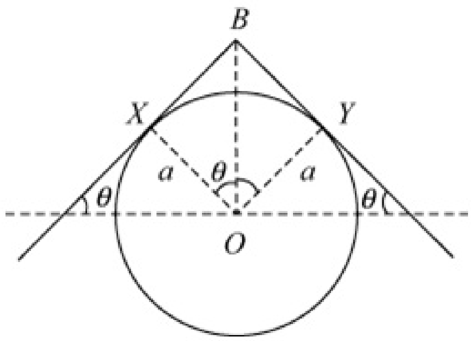


Hint: use the horizontal plane through the centre of the sphere as the zero level for the potential energy.

b Show that the rods are in equilibrium if $\cos^3\theta = \sin\theta$ and verify that $\theta = 0.60$ is accurate as a solution to 2 s.f.

Solution:

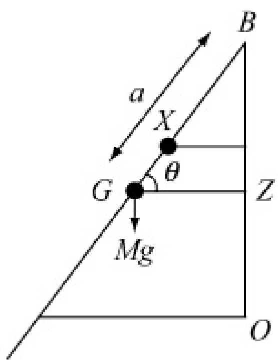
a



Take the horizontal level through the centre of the sphere O as the zero level for potential energy. Let the rods touch the sphere at points X and Y.

From geometry $X\hat{O}B = Y\hat{O}B = \theta$. ($O\hat{X}B = O\hat{Y}B = 90^\circ$ angle between tangent and radius.)

$\therefore BO = \dfrac{a}{\cos\theta} = a\sec\theta$



Consider one of the rods. Let its mid-point be G.

Then potential energy of rod $= Mg \times OZ$.

But $OZ = OB - BZ$

$= a\sec\theta - a\sin\theta$

$\therefore$P.E. of rod $= Mg(a\sec\theta - a\sin\theta)$

As there are two symmetric rods in the system

$V = 2Mg\left(a\sec\theta - a\sin\theta\right)$

[The constant here is zero but if you chose the base of the sphere as the zero level for P.E. then you would have a constant $2Mga$.]

b For equilibrium $\dfrac{dV}{d\theta} = 0$

But $\dfrac{dV}{d\theta} = +2Mga\sec\theta\tan\theta - 2Mga\cos\theta$

$\therefore 2Mga \times \dfrac{1}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta} = 2Mga\cos\theta$

$\therefore \sin\theta = \cos^3\theta$

If 0.60 is accurate to 2 s.f. there should be a sign change when substituting 0.595 and 0.605 into $f(\theta) = \sin\theta - \cos^3\theta$

$f(0.595) = -7.46 \times 10^{-3} < 0$

$f(0.605) = 0.012 > 0$

Sign change $\therefore$ 0.60 is a solution accurate to 2 s.f.

Solutionbank M4

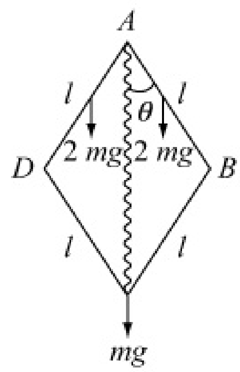
Edexcel AS and A Level Modular Mathematics

Stability

Exercise A, Question 5

Question:

Four light rods each of length l are freely hinged at their ends to form a rhombus $ABCD$ which is suspended from point A.
A light spring of natural length l and modulus of elasticity $10mg$ connects the points A and C.
A particle of mass m is attached at point C and the rods AB and AD each carry a particle of mass $2m$ at their mid-points. C moves freely in a vertical line through A and the angle between AB and the downward vertical is θ.

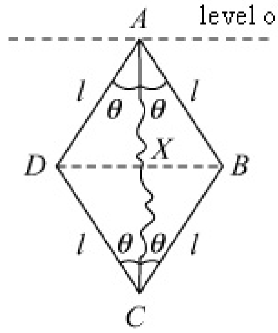


a Show that the potential energy of the system V is given by
$V = mgl(20\cos^2\theta - 24\cos\theta) + \text{constant}$.

b Find the values of θ which correspond to positions of equilibrium.

c Determine whether these values correspond to stable or to unstable equilibrium.

Solution:

a 

level of zero potential energy.

Take the horizontal through A as the zero level for potential energy.
Use symmetry to mark all the equal angles in the figure.
Let the diagonals meet at X.
From the isosceles $\Delta ADC, \Delta ADX$ is right-angled
$\therefore AX = l\cos\theta \Rightarrow AC = 2l\cos\theta$

$\therefore$ Extension x of the elastic string $AC = 2l\cos\theta - l$
The total P.E. of the system is V where

$$V = -2mg\frac{l}{2}\cos\theta - 2mg\frac{l}{2}\cos\theta - mg(AC) + \frac{1}{2}\lambda\frac{x^2}{l}$$

i.e. $V = -mgl\cos\theta - mgl\cos\theta - 2mgl\cos\theta + 5mg\dfrac{(2l\cos\theta - l)^2}{l}$

$$= -4mgl\cos\theta + 5mgl(4\cos^2\theta - 4\cos\theta + 1)$$
$$= mgl(20\cos^2\theta - 24\cos\theta) + \text{constant}$$

b $\dfrac{dV}{d\theta} = mgl\left[-40\cos\theta\sin\theta + 24\sin\theta\right]$

Put $\dfrac{dV}{d\theta} = 0$, then $8(3\sin\theta - 5\sin\theta\cos\theta) = 0$

i.e. $8\sin\theta(3 - 5\cos\theta) = 0$

$\therefore \sin\theta = 0$ or $\cos\theta = \dfrac{3}{5}$

$\therefore \theta = 0$ or 0.93 radians (2 s.f.)

c As $\dfrac{dV}{d\theta} = mgl\left[-20\sin 2\theta + 24\sin\theta\right]$

$$\frac{d^2V}{d\theta^2} = mgl\left[-40\cos 2\theta + 24\cos\theta\right]$$

when $\theta = 0, \dfrac{d^2V}{d\theta^2} = -16mgl < 0$ $\therefore$ unstable equilibrium

when $\theta = 0.93^c$ $\dfrac{d^2V}{d\theta^2} = mgl\left[-40\times\dfrac{-7}{25} + 24\times\dfrac{3}{5}\right]$

$$= \frac{128}{5}mgl > 0 \therefore \text{stable equilibrium.}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 6

Question:

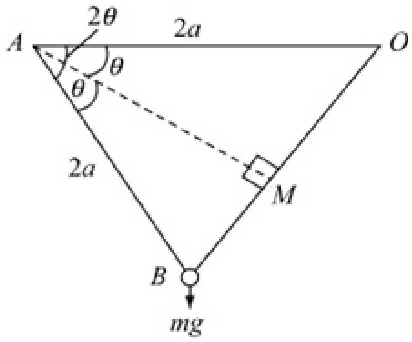
A light rod AB of length $2a$ can turn freely in a vertical plane about a smooth fixed hinge at A. A particle of mass m is attached at point B. One end of a light elastic string, of natural length $\frac{3}{2}a$ and modulus of elasticity $mg\sqrt{3}$ is also attached to the rod at B. The other end of the string is attached to a fixed point O at the same horizontal level as A. Given that $OA = 2a$ and that the angle between AB and the horizontal is 2θ,

a show that, provided the string remains taut, the potential energy of the system is given by $V = -2mga(\sin 2\theta + \frac{4}{3}\sqrt{3}\cos 2\theta + 2\sqrt{3}\sin\theta) + \text{constant}$.

b Verify that there is a position of equilibrium in which $\theta = \frac{\pi}{6}$ and determine the stability of this equilibrium.

Solution:

a



……level of zero potential energy.

$$\text{P.E. of particle} = -mg \times 2a \sin 2\theta$$

$$\text{P.E. of string} = \frac{1}{2} mg\sqrt{3}\frac{x^2}{\frac{3}{2}a} = \frac{1}{3}mg\sqrt{3}\frac{x^2}{a}$$

But from the isosceles triangle OAB length $OB = 2 \times BM = 2 \times 2a \sin \theta$

$\therefore$ Extension $x = 4a \sin \theta - \frac{3a}{2}$

$$\therefore \text{Total P.E., } V = -2mga \sin 2\theta + \frac{1}{3}mg\sqrt{3}a\left[4\sin\theta - \frac{3}{2}\right]^2$$

$$\begin{aligned} \text{i.e. } V &= -2mga\left[\sin 2\theta - \frac{1}{6}\sqrt{3}\left(16\sin^2\theta - 12\sin\theta + \frac{9}{4}\right)\right] \\ &= -2mga\left[\sin 2\theta - \frac{1}{6}\sqrt{3}\left(8 - 8\cos 2\theta - 12\sin\theta + \frac{9}{4}\right)\right] \\ &= -2mga\left[\sin 2\theta + \frac{4}{3}\sqrt{3}\cos 2\theta + 2\sqrt{3}\sin\theta\right] + \text{constant} \end{aligned}$$

b $\frac{dV}{d\theta} = -2mga\left[2\cos 2\theta - \frac{8}{3}\sqrt{3}\sin 2\theta + 2\sqrt{3}\cos\theta\right]$

When $\theta = \frac{\pi}{6}, \frac{dV}{d\theta} = -2mga\left[2\cos\frac{\pi}{3} - \frac{8}{3}\sqrt{3}\sin\frac{\pi}{3} + 2\sqrt{3}\cos\frac{\pi}{6}\right]$

$= -2mga\,[1 - 4 + 3] = 0$

This confirms that $\theta = \frac{\pi}{6}$ gives a position of equilibrium.

$\frac{d^2V}{d\theta^2} = -2mga\left[-4\sin 2\theta - \frac{16}{3}\sqrt{3}\cos 2\theta - 2\sqrt{3}\sin\theta\right]$

when $\theta = \frac{\pi}{6}, \frac{d^2V}{d\theta^2} = -2mga\left[-2\sqrt{3} - \frac{8}{3}\sqrt{3} - \sqrt{3}\right] = \frac{34}{3}\sqrt{3}mga > 0$

$\therefore$ this is a position of stable equilibrium.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 7

Question:

A small bead B of mass km can slide on a smooth vertical circular wire with centre O and radius a which is fixed in a vertical plane. B is attached to one end of a light elastic string of natural length $\frac{3}{2}a$ and modulus of elasticity $12mg$. The other end of the string is attached to a fixed point A which is vertically above the centre point O of the circular wire.
The angle between the string AB and the downward vertical at A is θ.

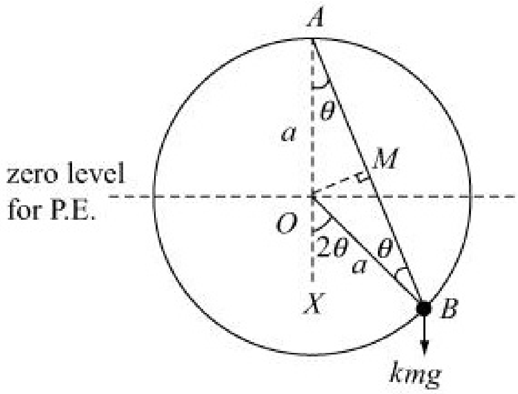
a Show that the potential energy V of the system is given by
$V = 2mga((8-k)\cos^2\theta - 12\cos\theta) + \text{constant}$.

b Find the restrictions on k if there is only one point of equilibrium, where $\theta = 0$.

c Subject to these restrictions, determine the stability of this equilibrium.

Solution:

a



Note $B\hat{O}X = 2\theta$

(angle at centre $= 2\times$ angle at circumference)

As ΔAOB is isosceles

$O\hat{B}A = O\hat{A}B = \theta$

Also $AB = 2\times AM$, where M is the mid-point of AB.

So $AB = 2\times a\cos\theta = 2a\cos\theta$

Let the extension in the string be x.

Then $x = 2a\cos\theta - \frac{3a}{2}$

The potential energy of the bead $B = -kmga\cos 2\theta$

The potential energy of the string $= \frac{1}{2}\times 12mg\frac{x^2}{3\frac{a}{2}} = 4mg\frac{a^2}{a}\left(2\cos\theta - \frac{3}{2}\right)^2$

$\therefore$ Total potential energy $V = -kmga\cos 2\theta + 4mga\left(4\cos^2\theta - 6\cos\theta + \frac{9}{4}\right)$

i.e. $V = -kmga(2\cos^2\theta - 1) + 16mga\cos^2\theta - 24mga\cos\theta + 9mga$

$= 2mga((8-k)\cos^2\theta - 12\cos\theta) + \text{constant}$

b $\frac{dV}{d\theta} = 2mga\left[-2(8-k)\cos\theta\sin\theta + 12\sin\theta\right]$

Put $\frac{dV}{d\theta} = 0 \;\therefore 4mga\sin\theta\left[6-(8-k)\cos\theta\right] = 0$

$\therefore \sin\theta = 0$ *or* $\cos\theta = \frac{6}{8-k}$

Only one point of equilibrium if $\frac{6}{8-k} \geq 1$ i.e. $2 \leq k < 8$

c $\frac{d^2V}{d\theta^2} = 2mga\left[-2(8-k)\cos 2\theta + 12\cos\theta\right]$

When $\theta = 0$

$\frac{d^2V}{d\theta^2} = 2mga\left[-2(8-k) + 12\right]$

$= 2mga\left[2k-4\right] \geq 0$ as $k \geq 2$

$\therefore$ Equilibrium is stable.

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Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 8

Question:

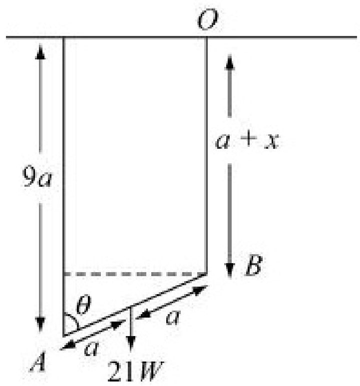
A uniform rod AB of length $2a$ and weight $21W$ is freely pivoted to a fixed support at A. A light elastic string of natural length a and modulus $\frac{3}{2}W$ has one end attached to B and the other to a small ring which is free to slide on a smooth horizontal straight wire passing through a point at a height $9a$ above A.

a Show that when the rod makes an angle θ with the upward vertical at A and the string is vertical, the potential energy of the system is $V = 3Wa\cos\theta(\cos\theta - 1) + \text{constant}$.

b Find the positions of equilibrium and determine whether they are stable or unstable.

Solution:

a



zero level for P.E.

P.E. of rod $= -21W\left(9a - a\cos\theta\right)$

P.E. of string $= \frac{1}{2} \times \frac{3}{2} W \frac{x^2}{a}$

where $a + x = 9a - 2a\cos\theta$

$\therefore x = 8a - 2a\cos\theta$

So total P.E. $V = -21W\left(9a - a\cos\theta\right) + \frac{3W}{4a} a^2 \left(8 - 2\cos\theta\right)^2$

i.e. $V = -189Wa + 21Wa\cos\theta + \frac{3}{4}Wa(64 - 32\cos\theta + 4\cos^2\theta)$

$= 3Wa\cos^2\theta - 24Wa\cos\theta + 21Wa\cos\theta + 48Wa - 189Wa$

$\therefore V = 3Wa\cos\theta(\cos\theta - 1) + \text{constant}$

b $\frac{dV}{d\theta} = 3Wa\cos\theta(-\sin\theta) - 3Wa\sin\theta\left(\cos\theta - 1\right)$

Put $\frac{dV}{d\theta} = 0$, then $-6Wa\cos\theta\sin\theta + 3Wa\sin\theta = 0$

$\therefore \sin\theta\left(1 - 2\cos\theta\right) = 0$

i.e. $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\therefore \theta = 0, \pi$ or $\frac{\pi}{3}$

$\frac{d^2V}{d\theta^2} = -6Wa\cos 2\theta + 3Wa\cos\theta$

When $\theta = 0$ $\frac{d^2V}{d\theta^2} = -3Wa < 0$ $\therefore$ unstable

$\theta = \pi$ $\frac{d^2V}{d\theta^2} = -9Wa < 0$ $\therefore$ unstable

$\theta = \frac{\pi}{3}$ $\frac{d^2V}{d\theta^2} = 3Wa + \frac{3Wa}{2} > 0$ $\therefore$ stable

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 9

Question:

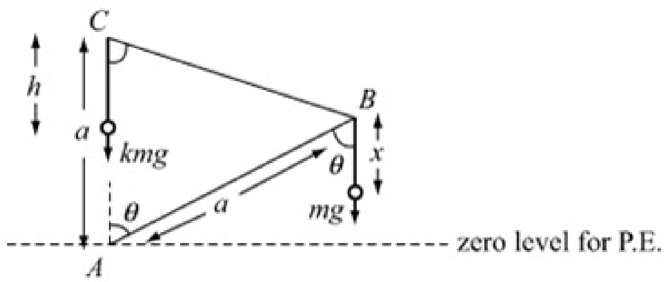
A light rod AB can freely turn in a vertical plane about a smooth hinge at A and carries a mass m hanging from B. A light string of length $2a$ fastened to the rod at B passes over a smooth peg at a point C vertically above A and carries a mass km at its free end. If $AC = AB = a$,

a find the range of values of k for which equilibrium is possible with the rod inclined to the vertical.

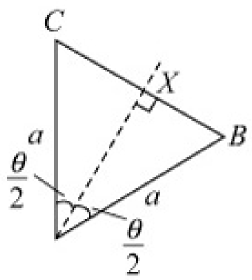
b Given that equilibrium is possible with the rod horizontal find the value of k.

c If the rod is slightly disturbed when horizontal and in equilibrium, determine whether it will return to the horizontal position or not. **[E]**

Solution:



a ΔABC is isosceles and $CB = 2 \times CX$ where



$CX \quad = a\sin\frac{\theta}{2}$

$\therefore CB = 2a\sin\frac{\theta}{2}$

As the string has length $2a$, $h = 2a - 2a\sin\frac{\theta}{2}$

$\therefore$ Total P.E. $V = +kmg\left(a - \left(2a - 2a\sin\frac{\theta}{2}\right)\right) + mg(a\cos\theta - x)$

where x is constant.

$\therefore V = 2kmga\sin\frac{\theta}{2} + mga\cos\theta + \text{constant}$

For equilibrium, $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = kmga\cos\frac{\theta}{2} - mga\sin\theta$$

$$= kmga\cos\frac{\theta}{2} - 2mga\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= mga\cos\frac{\theta}{2}\left(k - 2\sin\frac{\theta}{2}\right)$$

$\therefore$ Equilibrium when $\cos\frac{\theta}{2} = 0$ or when $\sin\frac{\theta}{2} = \frac{k}{2}$ when $\cos\frac{\theta}{2} = 0, \theta = \pi$,

i.e. not inclined to the vertical.

$\therefore \sin\frac{\theta}{2} = \frac{k}{2}$ must have a solution

As $0 < \sin\frac{\theta}{2} < 1$

$\therefore 0 < k < 2$

b When the rod is horizontal $\theta = \frac{\pi}{2}$.

$\therefore k - 2\sin\frac{\pi}{4} = 0$ for equilibrium

i.e. $k = \sqrt{2}$

c $\frac{d^2V}{d\theta^2} = -\frac{kmga}{2}\sin\frac{\theta}{2} - mga\cos\theta$

Substitute $\theta = \frac{\pi}{2}$ and $k = \sqrt{2}$

$\therefore \frac{d^2V}{d\theta^2} = -\frac{mga}{2} < 0$

$\therefore$ unstable so will not return to horizontal position.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Stability

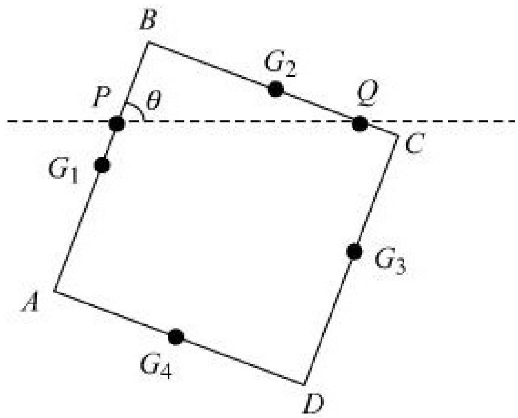
Exercise A, Question 10

Question:

Four equal uniform rods, each of length $2a$ and each of mass M are rigidly joined together to form a square frame. The frame hangs at rest in a vertical plane on two pegs P and Q which are at the same level as each other.
If $PQ = b$ and the pegs are each in contact with different rods, show that the potential energy V satisfies the equation $V = 2mg(b\sin 2\theta - 2a\sin\theta - 2a\cos\theta)$.
Find the three positions of equilibrium if $b = \sqrt{2}a$ and determine the stability of each of them.

Solution:

a



The horizontal through points P and Q is the zero level for potential energy.
(The mid-point of PQ will be vertically above the centre of the square.)
Label the square $ABCD$.
Let θ be the angle between AB and the horizontal.
Let G_1, G_2, G_3 and G_4 be the mid-points of the four rods as shown.

$BP = b\cos\theta$

$\therefore PG_1 = (a - b\cos\theta)$

$\therefore$ Potential Energy of rod $AB = -Mg\,(a - b\cos\theta)\sin\theta$

Similarly

$BQ = b\sin\theta$, and so $G_2Q = (b\sin\theta - a)$

$\therefore$ Potential Energy of rod $BC = Mg\,(b\sin\theta - a)\cos\theta$

Potential Energy of rod $CD = -Mg\,(2a - b\sin\theta)\cos\theta - Mga\sin\theta$ and

Potential Energy of rod $AD = -Mg\,(2a - b\cos\theta)\sin\theta - Mga\cos\theta$

$\therefore$ Total Potential Energy

$$\begin{aligned} V &= -Mga\sin\theta + Mgb\cos\theta\sin\theta \\ &\quad + Mgb\cos\theta\sin\theta - Mga\cos\theta \\ &\quad - Mga\sin\theta + Mgb\cos\theta\sin\theta - 2Mga\cos\theta \\ &\quad - 2Mga\sin\theta + Mgb\sin\theta\cos\theta - Mga\cos\theta \end{aligned}$$

i.e. $V = -4Mga\sin\theta + 4Mgb\sin\theta\cos\theta - 4Mga\cos\theta$

$$= 2Mg\,[b\sin 2\theta - 2a\sin\theta - 2a\cos\theta]$$

If $b = \sqrt{2}a$

$$V = 2\sqrt{2}Mga\left[\sin 2\theta - \sqrt{2}\sin\theta - \sqrt{2}\cos\theta\right]$$

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = 2\sqrt{2}Mga\left[2\cos 2\theta - \sqrt{2}\cos\theta + \sqrt{2}\sin\theta\right]$$

Put $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$

Then $2\cos 2\theta - \sqrt{2}\,(\cos\theta - \sin\theta) = 0$

$\therefore 2\left(\cos^2\theta - \sin^2\theta\right) - \sqrt{2}\,(\cos\theta - \sin\theta) = 0$

$\therefore \sqrt{2}(\cos\theta - \sin\theta)\left[\sqrt{2}(\cos\theta + \sin\theta) - 1\right] = 0$

$$\therefore \cos\theta = \sin\theta \quad \text{or} \quad \cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$$

$$\text{i.e. } \tan\theta = 1 \quad \text{or} \quad \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\text{i.e. } \theta = \frac{\pi}{4} \quad \text{or } \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{\pi}{3} + \frac{\pi}{4} \quad \text{or} \quad \theta = -\frac{\pi}{3} + \frac{\pi}{4}$$

$$\text{i.e. } \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{7\pi}{12} \quad \text{or} \quad \theta = \frac{-\pi}{12}$$

$$\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\left[-4\sin 2\theta + \sqrt{2}\sin\theta + \sqrt{2}\cos\theta\right]$$

when $\theta = \frac{\pi}{4}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\,[-4+1+1] = -4\sqrt{2}Mga < 0$ $\therefore$ unstable.

when $\theta = \frac{7\pi}{12}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\,[2+1] = 6\sqrt{2}Mga > 0$ $\therefore$ stable.

when $\theta = -\frac{\pi}{12}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\left[+2 + \frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{\sqrt{3}}{2}\right]$

$$= 6\sqrt{2}Mga > 0 \;\therefore \text{ stable}$$